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Progress was made along several fronts during this period. Classes of RBF, periodic, and nonstationary spherical wavelets were constructed, all being capable of synthesizing and analyzing scattered data. Shape preservation problems were investigated. A framework for obtaining rates of approximation in non-traditional settings and data – i.e. compact manifolds and generalized Hermite data, was provided. This framework was used to derive approximation rates for various classes of functions on the circle and the 2-sphere. In particular, for a broad class of functions, rates of approximation were obtained for scattered data on the m-sphere. Stability questions related to these classes of functions were studied. Results from the work above were applied to neural networks.

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"Surface Fitting via Radial and Related Basis Functions with Applications to Neural Networks"

Period: 1 March 1995—28 February 1998

by

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25 February 1998

Summary

The following is a summary of the research of F. J. Narcowich and J. D. Ward supported by the Air Force during the period 1 March 1995 through 28 February 1998. Progress was made along several fronts during this period. Classes of RBF, periodic, and nonstationary spherical wavelets were constructed, all being capable of synthesizing and analyzing scattered data. Shape preservation problems were investigated. A framework for obtaining rates of approximation in non-traditional settings and data–i.e. compact manifolds and generalized Hermite data–, was provided. This framework was used to derive approximation rates for various classes of functions on the circle and the 2-sphere. In particular, for a broad class of functions, rates of approximation were obtained for scattered data on the m-sphere. Stability questions related to these classes of functions were studied. Results from the work above were applied to neural networks.

1 Review of Research

The research described below was carried out during the period 1 March 1995 to 28 February 1998. The work itself is communicated in the papers and informal technical reports listed in §2. Publications are denoted by 'P', technical reports

by 'T'. The numbering is from the lists in §2.1 and §2.2. Our research fits into four broad categories described below.

1.1 Wavelets

In [P2], we constructed orthogonal wavelets on the scaled lattice of $\mathbf{R}$ employing an RBF for a scaling function; this was the first time RBFs had been used to in this way. The resulting orthonormal wavelets possessed good to very good time-frequency localization (depending on the given RBF) and, in addition, satisfied the Littlewood-Paley identity. Moreover, the approach taken in [P1] can be extended to one for constructing wavelets based on ***non-equally*** spaced translates of a given RBF, in one or several dimensions. This we did in [P1], where we provided a construction of wavelets generated by translates of a given tempered distribution h; the translates are defined by a sequence of arbitrarily spaced points. Our approach provides not only a unifying thread for the known results, but also a very powerful tool for analyzing almost any mathematical representation of scattered discrete (univariate) data. For example, our methods yield the first results on stability of certain wavelet bases associated with scattered data on the line under the sole assumption that the scattered data are taken on sample sites with bounded global mesh ratio. An important result of this study is an exact identification of the $L^2(\mathbf{R})$-subspaces generated by linear combinations of shifts of certain unbounded radial basis functions.

In [P12], we investigated a class of nonstationary, orthogonal, periodic scaling functions and wavelets generated by continuously differentiable periodic functions with positive Fourier coefficients; these are just *periodic basis functions* (PBFs) for the circle. The scaling functions and wavelets presented there have a number of attractive features. The decomposition and reconstruction coefficients involve only a few terms and they can be computed via FFT. To discuss the localization properties, we adapted a quantum-mechanical uncertainty relation for the circle to the case at hand, and we gave a class of wavelets that are well localized and can be as smooth as one desires. The idea is that using related bases to do both representation and analysis will provide an integrated approach to handling periodic data, even when the data are scattered or noisy. Another possible use for these wavelets is detection of damage to a neural network. Often a network will compensate for damage, and it is sometimes difficult to know when and where such damage has occurred. In the case of neural beamforming, one can use the PBF wavelets, which are naturally periodic, to continuously monitor the network to detect derivative discontinuities that would arise from damage to the network.

In [P11, P13], we developed classes of nonstationary wavelets generated by a *spherical basis functions* (SBFs), which comprise a subclass of Schoenberg's positive definite spherical functions [6]. In particular, these wavelets are intrinsically defined on the m-sphere and are independent of the coordinate system. Thus such wavelet families do not have to treat the north and south pole dif-

ferently, as is the case with most other families of spherical wavelets.

We wish to point out one very significant result we obtained in [P11, P13]: namely, a new "uncertainty principle" that explicitly measures the trade-off between smoothness and localization for a function on the 2-sphere. We discussed the smoothness/localization tradeoff for one class of spherical wavelets that we constructed, and it also has been used and adapted by Freeden and Windheuser [3] for a different class of spherical wavelets.

Recently, these authors were able to construct locally supported (i.e. the support is in some spherical patch) positive definite functions on the sphere and thus the autocorrelation matrices used in the construction of such wavelets can be made banded. These locally supported functions are "radialized" univariate splines restriction (when viewed as functions on R^3) to S^2. It is hoped that further understanding of SBFs might lead to efficient algorithms for these wavelets.

1.2 Shape Preservation

How to provide a fit to data while preserving some feature that the data is known to have is–increasing, convex, positive, etc.–is an important one. The papers [P3, P4] attack this problem.

Paper [P4] continues the study of best approximation in a Hilbert space X from a subset K which is the intersection of a closed convex cone C and a closed linear variety, with special emphasis on applications to the n-convex functions. It was shown in a previous paper that finding best approximations in K to any function in X can be reduced to the (generally much easier) problem of finding best approximations to a certain perturbation of f from either the cone C or a certain subcone C_F. We show how to determine this subcone C_F, give the precise condition characterizing when $C_F = C$, and apply and strengthen these general results in the practically important case when C is the cone of n-convex functions in $L^2(a, b)$.

The main point of [P3] was to show that $P_K(x)$ is equivalent to determining $P_C(x + A^*y)$ –the best approximation to a certain perturbation $x + A^*y$ of x – from the convex set C or from a certain subset C_b of C. Previous to this, the result had only been known in the case of a convex cone or for *special* data sets associated with a closed convex set. Moreover, in many cases the best approximation $P_C(x + A^*y)$ can be obtained from existing algorithms.

1.3 Rates of Approximation and Stability

In [P5,P6] Dyn, Narcowich, and Ward provided a framework for obtaining rates of approximation in non-traditional settings and data–i.e., compact manifolds and generalized Hermite data—, and used this framework to derive approximation rates for various classes of functions on the circle and the 2-sphere. Earlier, Freeden, Schreiner, and Franke [2] had obtained linear rates for scattered-data

point-interpolation on the 2-sphere. The rates that we obtained in [P6] were substantially better than linear. They were also for point-interpolation, and they depended on the function class, the SBF and the set of (gridded) points chosen. These rates were the first to reflect both the smoothness of the function to be approximated and the smoothness of the approximating function. For the circle, the rates in [P6] were for a limited class of Hermite problems, but included the case of quasi-uniform, scattered data.

Jetter, Stöckler and Ward introduced a new technique in [P8, T1] involving *norming sets* and used them in conjunction with the methods in [P6] to obtain rates for scattered-data point interpolation on spheres.

For the multilevel interpolation method first used in connection with RBFs in [1], Narcowich, Schaback, and Ward provided in [T2] a framework with which to analyze rates, and in addition they did so in specific cases. This work was the first to analyze rates in RBF cases where the parameters (spread, type of RBF, etc.) were allowed to vary.

1.4 Neural Networks

Much of our work is connected with improving the neural beamforming algorithm described in [7, 8]. In particular, the driving force behind our constructing and studying the RBF wavelets in [P1, P2] and the periodic wavelets in [P12], the idea being to combine synthesis and analysis using a unified approach involving PBFs. We also have written three informal AFOSR technical reports [T3, T4, T5], all specifically dealing with neural networks.

In [T4], we discuss the neural beamforming method introduced in [7] and described in detail in [8]. We show that this method involves a *local* form of Lagrange interpolation, discuss the mathematics behind it, and suggest improvements to the algorithm. In [T3], we introduce a new neural beamforming method for detecting a single source via an array of $m+1$ antennas; the method fits a residual signal, rather than the original signal. This method works well in smoothing noisy signals. Both methods employ the periodic basis functions introduced by us in [5].

Dr. H. Southall and (Rome Laboratory, Hanscom AFB) observed that the neural net for an ideal phased array antenna produces output very similar to a Butler matrix, and, with Dr. R. Mailloux (Rome Laboratory, Hanscom AFB), wrote a paper [4] comparing the two. Dr. Southall asked us what the mathematical basis for this phenomenon was.

The report [T5] provides an interesting answer to this question. Viewed from a digital perspective, the Butler matrix is a feed-forward neural net that uses what amounts to a PBF as an activation function. We showed that near the center of one of its orthogonal beams, the network output is independent of which PBF one uses as an activation function!

2 Publications, Reports, and Talks

2.1 Publications

1. C. K. Chui, K. Jetter, J. Stöckler and J. D. Ward, "Wavelets for analyzing scattered data: An unbounded operator approach," *Applied and Computational Harmonic Analysis*, **3**, (1996), 254-267.

2. C. K. Chui, J. Stöckler and J. D. Ward, "Analytic wavlets generated by radial functions," *Advances in Computational Mathematics*, **5**, (1996), 95-123.

3. F. Deutsch, W. Li and J. D. Ward, "A dual approach to constrained interpolation from a convex subset of Hilbert space," *J. of Approx. Theory*, **90**, (1997), 385-414.

4. F. Deutsch, V. Ubhaya, J. D. Ward and Y. Xu, "Constrained best L_2-approximation by n-convex functions," *Constructive Approximation*, **12**, (1996), 361-384.

5. N. Dyn, F. J. Narcowich and J. D. Ward, "A framework for interpolation and approximation on Riemannian manifolds," *Approximation Theory and Optimization*, Ed. M. Buhmann and A. Iserles, Cambridge University Press (1997).

6. N. Dyn, F. J. Narcowich and J. D. Ward, "Variational principles and Sobolev-type estimates for generalized interpolation on a Riemannian manifold," *Constructive Approximation*, to appear.

7. T. N. T. Goodman, C. A. Micchelli and J. D. Ward, "Spectral radius formulas for the Dilation-Convolution operator," *Bull. of the South East Asian Math Society*, **19**, (1995), 95-106.

8. K. Jetter, J. Stöckler and J. D. Ward, "Error estimates for scattered data interpolation on spheres," *Constructive Approximation*, to appear.

9. F. J. Narcowich, N. Sivakumar and J. D. Ward, "Stability results for scattered-data interpolation on Euclidean spheres," *Adv. Comput. Math.*, to appear.

10. F. J. Narcowich, P. W. Smith, and J. D. Ward, "Density of Translates of Radial Functions on Compact Sets," in Proceedings of the *Eighth International Conference on Approximation Theory*, Ed. by Charles K. Chui and Larry L. Schumaker, World Scientific, 435-442, 1995.

11. F. J. Narcowich and J. D. Ward, "Nonstationary Spherical Wavelets for Scattered Data," in Proceedings of the *Eighth International Conference on Approximation Theory*, Ed. by Charles K. Chui and Larry L. Schumaker, World Scientific, 301-308, 1995.

12. F. J. Narcowich and J. D. Ward, "Wavelets associated with periodic basis functions," *Applied and Computational Harmonic Analysis*, **3**, (1996), 40-56.

13. F. J. Narcowich and J. D. Ward, "Nonstationary wavelets on the m-sphere for scattered data," *Applied and Computational Harmonic Analysis*, **3**, (1996), 324-336.

2.2 Technical Reports

1. K. Jetter, J. Stöckler and J. D. Ward, "Norming sets and spherical cubature formulas."Center for Approximation Theory Report # 382, Department of Mathematics, Texas A & M University, 1998.

2. F. J. Narcowich, R. Schaback and J. D. Ward, "Multilevel interpolation and approximation," Center for Approximation Theory Report # 379, Department of Mathematics, Texas A & M University, 1997.

3. F. J. Narcowich and J. D. Ward, "The Method of Residuals," May 17, 1996. (Informal report.)

4. F. J. Narcowich and J. D. Ward, "Neural Beamforming and Lagrange Interpolation," June 6, 1996. (Informal report.)

5. F. J. Narcowich and J. D. Ward, "The Butler/Neural-Net Analogy," January 29, 1997. (Informal report.)

2.3 Research Conference Talks

F. J. Narcowich

- **Half-hour contributed talk.** "Nonstationary Spherical Wavelets for Scattered Data," **Eighth International Conference on Approximation Theory**, College Station, TX, January 8-12, 1995.

- **Half-hour invited talk.** "Spherical basis functions and intrinsic spherical wavelets," Session on **Geodesy and Approximation Theory, Oberwolfach**, October 4, 1995.

- **Half-hour invited talk.** "The Multilevel Method: Rates of Approximation," **1997 IMACS Conference on Radial Basis Functions** , May 27-30, 1997, Monterey, CA.

- **One-hour plenary address.** "Recent Developments in Approximation via Positive Definite functions," **Ninth International Conference on Approximation Theory**, Nashville, TN, January 3–6, 1998.

J. D. Ward

- **Half-hour talk.** "Density of Translates of Radial Functions on Compact Sets," **Eighth International Conference on Approximation Theory**, College Station, TX, January 8-12, 1995.
- **Half-hour invited talk. International conference on scattered data fitting**, Cancun, Mexico, March 1995.
- **Half-hour invited talk.** "Nonstaionary wavelets on the m-sphere, localization and uncertainty," Session on **Geodesy and Approximation Theory, Oberwolfach**, October 4, 1995.
- **One-hour invited talk. International Workshop on Wavelets**, University of Montreal, February 1996.
- **Half-hour invited talk. International Conference on Curves and Surfaces**, Chamonix, France, June 1996.
- **One-hour invited talk.** Session on **Numerical Methods in Approximation Theory, Oberwolfach**, May 12-16, 1996.
- **One-hour invited talk.** University of Hohenheim (Stuttgart) May 21, 1997.
- **Half-hour invited talk.** Special session on **Linear Programming in Approximation and Algorithms, Informs**, San Diego, May 6, 1997.

3 Activities

In addition to the publications, reports, and talks listed in §2, we have had consistent contact with Dr. Hugh Southall of Rome Laboratory, Hanscomb AFB. During the period of this report, Dr. Southall visited us here at Texas A&M University twice, in in December 1995 and December 1996. We paid visits to the laboratory at Hanscom AFB in July 1995, June and August 1996, and in July 1995 and August 1997. In January 1998, all of us met at Ninth International conference on Approximation Theory held in Nashville. Finally, we have exchanged numerous messages, via both surface and electronic mail, concerning questions related to work being carried out in Dr. Southall's laboratory at Hanscom AFB.

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[2] W. Freeden, M. Schreiner, and R. Franke, "A Survey on Spherical Spline Approximation," *Surveys Math. Indust.*, **7** (1997), 29-85.

[3] W. Freeden and U. Windheuser, "Combined spherical harmonical and wavelet expansion," Berichte der Arbeitsgruppe Technomathematik von Universität Kaiserlautern # 144, 1995.

[4] R. J. Mailloux and H. L. Southall, "The analogy between the Butler matrix and the neural network direction finding array," preprint.

[5] F. J. Narcowich, "Generalized Hermite Interpolation and Positive Definite Kernels on a Riemannian Manifold," *J. Math. Anal. Applic.* **190** (1995), 165-193.

[6] I. J. Schoenberg, "Positive Definite Functions on Spheres," *Duke Math. J.* **9** (1942), 96-108.

[7] J. Simmers, H. L. Southall, and T. O'Donnell, "Advances in Neural Beamforming", in *Proceedings of the 1993 Antenna Applications Symposium (Robert Allerton Park), University of Illinois*, Griffiss AFB, NY, September, 1993, USAF HQ Rome Laboratory.

[8] H. L. Southall, J. A. Simmers, and T. H. O'Donnell, "Direction Finding in Phased Arrays with a Neural Network Beamformer", *IEEE Transactions on Antennas and Propagation*, vol. 43, no. 12, pp. 1369–1374, December 1995.